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An analytic solution for the potential due to a circular parallel plate capacitor

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Abstract. An analytic solution is given for the potential due to a capacitor consisting of two coaxial circular discs with equal radii and of infinitesimal thickness. The potential is expressed as a rapidly converging infinite series which can be readily evaluated. The utility of the expression is illustrated by two examples in which the potential has been computed.

1. Introduction

Our interest in the electrostatic potential due to a circular parallel plate (button) capacitor arose while investigating the heating of tumours by passing electrical currents between implanted electrodes for the purpose of cancer therapy (hyperthermia). Love (1949) investigated the problem of the button capacitor to take into account the fringing electric field in an apparatus for e/m measurements. More recently, Hutson (1975) published a paper on the potential due to various electrode configurations. The final objective of his work was the heating of body tissues, similar to our own effort.

Methods for computing the electrostatic potential due to a conducting circular plate of infinitesimal thickness, charged to a given potential, can be found in many textbooks (see e.g. Jackson 1962, Titchmarsh 1967, Tranter 1965). The problem was first solved by Weber (see e.g. Jackson 1962) who represented the potential by a Hankel transform. Guided by a study of numerous integrals involving Bessel functions he 'spotted' the proper kernel which is, in general, very difficult to obtain. Later, the researchers Tranter (1965), Titchmarsh (1967) and Copson (1947) suggested various systematic procedures which could have been used to find the kernel in that example. Unfortunately, these procedures cannot be readily adapted to the problem of finding the potential due to a capacitor consisting of two oppositely charged circular discs. The difficulties are due to the fact that each of the suggested systematic methods requires a certain knowledge about the potential or some of its derivatives throughout the entire coordinate space. While it is possible to obtain that knowledge in selected simple problems (for example from symmetry considerations), it would be most difficult to obtain for the parallel plate capacitor.

Nicholson (1924) gives an expression for the electric potential due to the parallel plate capacitor as an infinite series of spheroidal harmonics with coefficients satisfying an infinite set of linear equations. As pointed out by Love (1949), this solution does not converge for all diameter-to-plate separation ratios. Love gives a converging although implicit expression for the potential. The final expression he obtains is an integral transform involving a function which is represented as an infinite series. Each term in the series contains an integral for which the author does not give an evaluation. The numerical method used by Hutson (1975) should be adaptable for computing the potential due to the button capacitor. Being based on a finite difference method, however, the program cannot be expected to take properly into account the peculiar behaviour of the potential at the edges of the electrodes. The gradient of the potential (the electric field) goes to infinity as the electrode edges are approached. In addition, Hutson does not give a detailed description of the program which would make it readily usable for other researchers.

In the present paper we give an analytic solution for the potential due to the circular parallel plate capacitor. We represent the potential as a Hankel transform and 'spot' the proper kernel by studying numerous integrals involving Bessel functions, similar to the procedure used by Weber. The final expression can be readily evaluated and this is illustrated by computations of the potential distribution due to a capacitor with infinite spacing and another capacitor with finite spacing between the plates. We hope that the given solution will be of academic interest as well as useful in practical applications. Textbooks on electromagnetic theory demonstrate the application of the Hankel transform method on examples less familiar than the well known button capacitor. Jackson (1962), for instance, computes the potential due to an infinite conducting plate with a circular aperture. Smythe (1950) finds the potential due to a single, circular disc and that due to an infinite strip capacitor. Sommerfeld (1964) shows how the potential due to a semi-infinite strip capacitor can be obtained.

2. Solution for the electric potential

As shown in figure 1, the capacitor is assumed to consist of two circular disc electrodes of radius a. The electrodes are of infinitesimal thickness, placed a distance $2L$ apart, and are equally and oppositely charged to potentials $+V$ and $-V$. In computing the potential Φ we assume a coordinate system with its z axis passing through the centres of the two discs and with its xy plane coinciding with the median plane between the discs. Since there are no electrical charges anywhere except on the two discs, the

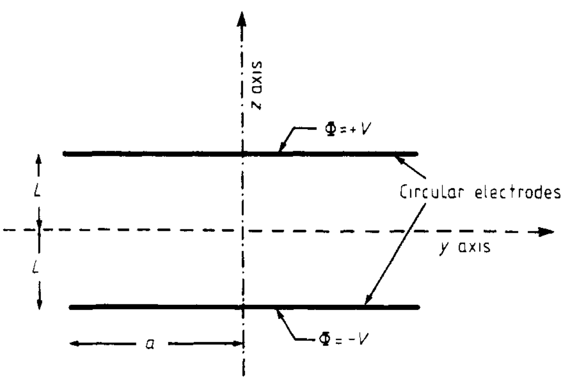

Figure 1. Parallel plate capacitor.

potential can be found as the solution of Laplace's equation subject to the appropriate boundary conditions. The axial symmetry simplifies Laplace's equation which can be written, in cylindrical coordinates,

$$\partial^2\Phi/\partial\rho^2 + \rho^{-1}\,\partial\Phi/\partial\rho + \partial^2\Phi/\partial z^2 = 0. \tag{1}$$

A solution to (1) finite at $\rho = 0$ is

$$\Phi(\rho, z) = \int_0^\infty A(k)(\mathrm{e}^{-k|z-L|} - \mathrm{e}^{-k|z+L|})J_0(k\rho)\,\mathrm{d}k, \tag{2}$$

where $J_0(k\rho)$ is the Bessel function of the first kind of order zero and $A(k)$ is a function which can be derived from the boundary conditions,

$$\Phi(\rho, \pm L) = \pm V, \qquad 0 \leqslant \rho \leqslant a. \tag{3}$$

We have chosen to write (2) using absolute values in the exponentials to ensure that the exponents will always be negative, a condition which is necessary in the evaluation of certain integrals. The proper function for $A(k)$ is

$$A(k) = (2V/\pi)(1 - \mathrm{e}^{-2kL})^{-1}(\sin ka)/k \tag{4}$$

and with this, (2) becomes

$$\Phi(\rho, z) = \frac{2V}{\pi}\int_0^\infty \frac{1}{1-\mathrm{e}^{-2kL}}\,\frac{\sin ka}{k}(\mathrm{e}^{-k|z-L|} - \mathrm{e}^{-k|z+L|})J_0(k\rho)\,\mathrm{d}k. \tag{5}$$

To show that (5) indeed satisfies the boundary conditions we note that at $z = \pm L$ the expression reduces to

$$\Phi(\rho, \pm L) = \pm\frac{2V}{\pi}\int_0^\infty \frac{\sin ka}{k}J_0(k\rho)\,\mathrm{d}k \tag{6}$$

and that the integral in (6) is a special case of Weber's discontinuous integrals (see e.g. Bowman 1958). The integral is evaluated as

$$\int_0^\infty \frac{\sin ka}{k}J_0(k\rho)\,\mathrm{d}k = \begin{cases} \tfrac{1}{2}\pi, & 0 \leqslant \rho \leqslant a, \\ \sin^{-1}(a/\rho), & \rho \geqslant a, \end{cases} \tag{7}$$

and with this, (6) reduces to

$$\Phi(\rho, \pm L) = \pm V, \qquad 0 \leqslant \rho \leqslant a.$$

Since solutions to Laplace's equation satisfying a specific set of boundary conditions are unique, (5) represents the desired potential due to the parallel plate capacitor.

In order to carry out the integration in (5) we expand the first term under the integral sign as

$$(1 - \mathrm{e}^{-2kL})^{-1} = \sum_{n=0}^{\infty} \mathrm{e}^{-2nkL} \tag{8}$$

and with this, (5) becomes

$$\Phi(\rho, z) = \frac{2V}{\pi}\sum_{n=0}^{\infty}\int_0^\infty \frac{\sin ka}{k}[\exp(-kb_{n-}) - \exp(-kb_{n+})]J_0(k\rho)\,\mathrm{d}k, \tag{9}$$

where

$$b_{n\pm} \equiv 2nL + |z \pm L|. \tag{10}$$

An evaluation of the integrals in (9) can be found in tables of integrals (see e.g. Gradshteyn and Ryzhik 1965); in particular,

$$\int_0^\infty \frac{\sin ka}{k} \exp(-kb_{n_\pm}) J_0(k\rho)\, dk = \sin^{-1}\frac{a}{x_{n_\pm}} \tag{11}$$

where

$$x_{n_\pm} \equiv \tfrac{1}{2}\{[b_{n_\pm}^2 + (\rho+a)^2]^{1/2} + [b_{n_\pm}^2 + (\rho-a)^2]^{1/2}\}. \tag{12}$$

With this, the potential due to the capacitor can be written as

$$\Phi(\rho, z) = \frac{2V}{\pi} \sum_{n=0}^{\infty} \left(\sin^{-1}\frac{a}{x_{n_-}} - \sin^{-1}\frac{a}{x_{n_+}}\right), \tag{13}$$

a form which is more amenable to evaluation than (5).

3. Discussion

As an example for the utility of (13) we compute the potential in the vicinity of the positively charged plate of a capacitor having large spacing between the plates. We use a coordinate system which is shifted along the z axis so that the positive plate lies in the xy plane of the new coordinate system, i.e. we use the transformation

$$z' = z - L, \qquad \rho' = \rho. \tag{14}$$

In the primed coordinate system we have

$$b_{n_-} = |z'| + 2nL, \qquad b_{n_+} = |z' + 2L| + 2nL, \tag{15}$$

and with this the potential can be written as

$$\begin{aligned}\Phi(\rho, z') = \frac{2V}{\pi}\bigg(&\sin^{-1}\frac{2a}{[(z')^2 + (\rho-a)^2]^{1/2} + [(z')^2 + (\rho+a)^2]^{1/2}} \\ &- \sin^{-1}\frac{2a}{[|z'+2L|^2 + (\rho-a)^2]^{1/2} + [|z'+2L|^2 + (\rho+a)^2]^{1/2}}\bigg) \\ &+ \frac{2V}{\pi}\sum_{n=1}^{\infty}\left(\sin^{-1}\frac{a}{x_{n_-}} - \sin^{-1}\frac{a}{x_{n_+}}\right).\end{aligned} \tag{16}$$

It can be seen that as $L \to \infty$ only the first term on the right-hand side will survive. This term can be recognised as the familiar expression for the potential due to a single disc. A similar calculation for the potential around the negatively charged disc yields the same type of result.

As a second illustration we computed equipotential surfaces due to a capacitor with $a/L = 2$ over the coordinate space $0 \leqslant \rho \leqslant 2a$, $0 \leqslant z \leqslant 2L$. The results, presented in figure 2, were obtained from a series truncated after 100 terms. Comparison with equipotential surfaces computed from a series with only 15 terms showed, due to the rapid convergence of the series, only minor discrepancies.

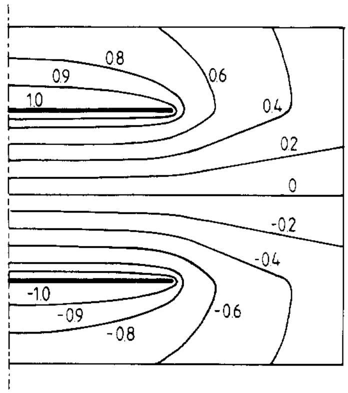

Figure 2. Lines of constant potential for a parallel plate capacitor with the electrodes charged to +1 and −1 V. Because of the symmetry only one half of the capacitor is shown.

4. Conclusion

An analytic solution for the potential due to a capacitor with parallel, circular plates has been found. It is represented by a rapidly converging infinite series which can be readily evaluated. As an example for the utility of our solution the potential in the vicinity of one of the electrodes was computed for a capacitor with infinite spacing between the plates. The infinite series was found to converge to the familiar simple expression for the potential due to a single charged disc. As a second example we graphically displayed the potential due to a capacitor, assuming the spacing between the plates to be equal to the plate radius.

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